MATH 6021 Lecture 3 9/21/2020

Recall: $\Sigma^{n-1} \subseteq \mathbb{R}^n$ min. hypersurface (immersed)

Stable $\iff$ $\displaystyle\int_\Sigma |A|^2\varphi^2 \leq \int_\Sigma |\nabla\varphi|^2 \quad \forall \varphi \in C_c^\infty(\Sigma)$

Bernstein Thm ($n=3$): Any entire min. graph in $\mathbb{R}^3$ is flat.

Stable Bernstein Conjecture: Any complete stable min. hypersurface in $\mathbb{R}^n$, $3 \leq n \leq 7$, is flat.

Recall counter e.g.'s from Bombieri - De Giorgi - Giusti

Fisher-Colbrie-Schoen '80, do Carmo-Peng '79:

The conjecture is true when $n=3$.

Remark: 1) FCS's proof also works with $\mathbb{R}^3$ replaced by (M^3, g) of non-negative scalar curvature.

2) The proof relies on a "vanishing theorem", which holds also in higher dimensions (recall: (M^n, g) closed, Ric $> 0 \Rightarrow H^1(M;\mathbb{R}) = 0$)

L^2-vanishing Theorem:

Let $\Sigma^{n-1} \subseteq \mathbb{R}^n$ be a complete (non-cpt) 2-sided stable min. hypersurface.

THEN, any 1-form $\omega \in \Omega^1(\Sigma)$ which is

(i) harmonic, i.e. $\Delta\omega = 0$ where $\Delta := d\delta + \delta d$

(ii) and in L^2, i.e. $\displaystyle\int_\Sigma |\omega|^2 < +\infty$.

must be identically zero, ie $\omega \equiv 0$.

Remark: This holds in all dimensions.

Proof: Key ideas: Bôchner technique & stability ineq.

Step 1: Use stability ineq. to prove a "weighted" stability ineq.

(1) $$\boxed{\int_\Sigma \big[|\omega|\, L(|\omega|)\big]\varphi^2 \le \int_\Sigma |\omega|^2 |\nabla\varphi|^2 \quad \forall\, \varphi \in C_c^\infty(\Sigma)}$$

$L(|\omega|)$ ← compute in Step 2

here: The Jacobi operator $L := \Delta_\Sigma + |A|^2$

Recall: Σ stable 2-sided $\Rightarrow \int_\Sigma |A|^2 \tilde{\varphi}^2 \le \int_\Sigma |\nabla\tilde{\varphi}|^2 \quad \forall\, \tilde{\varphi} \in C_c^\infty(\Sigma)$ (Lipchitz.)

Plug-in

Let $\omega \in \Omega^1(\Sigma)$ be an L^2 harmonic 1-form &

Set $f := |\omega| \ge 0$ Lip. Take $\tilde{\varphi} = f\varphi$ cpt. supp.

$$\Rightarrow \int_\Sigma |A|^2 f^2 \varphi^2 \le \int_\Sigma |\nabla(f\varphi)|^2 = \int_\Sigma |\nabla\varphi|^2 f^2 + |\nabla f|^2 \varphi^2 + \int_\Sigma 2\varphi f(\nabla f\cdot\nabla\varphi)$$

rewrite this →

Note: $$\int_\Sigma 2\varphi f(\nabla f\cdot\nabla\varphi) = \frac{1}{2}\int_\Sigma \nabla f^2 \cdot \nabla\varphi^2$$

$$\underset{\varphi\in C_c^\infty(\Sigma)}{\overset{\text{I.B.P.}}{=}} -\frac{1}{2}\int_\Sigma \varphi^2 \Delta f^2 = -\int_\Sigma \varphi^2 (f\Delta f + |\nabla f|^2).$$

Putting it back, we obtain

$$\int_\Sigma \underbrace{\underbrace{|A|^2 f^2\varphi^2 + f\Delta f\,\varphi^2}_{(f\Delta f + |A|^2 f^2)\varphi^2}}_{(fLf)\varphi^2} \le \int_\Sigma f^2 |\nabla\varphi|^2$$

recall: $f := |\omega|$

This finishes step 1.

Step 2: Compute $L(|\omega|)$ using Bôchner formula.

Recall: Bôchner formula for harmonic 1-forms ω on Σ

$$\frac{1}{2}\Delta(|\omega|^2) = |\nabla\omega|^2 + \underbrace{Ric_\Sigma(\omega^\#,\omega^\#)}_{\text{need to evaluate this!}}$$

We first rewrite the Ricci term using the Gauss eq^n.

Gauss eq^n: $R^\Sigma_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$ where $A = (h_{ij})$ 2^{nd} f.f. of Σ

($\mathbb{R}^n$ flat)

$$\xRightarrow[\text{over } j,l]{\text{trace}} \quad R^\Sigma_{ik} = h_{ik}\underbrace{\cancel{\sum_j h_{jj}}}_{=0\ \because\ \Sigma\text{ min}} - \sum_j h_{ij}h_{jk} \approx \underbrace{-A^2}_{\text{as an operator}}$$

Locally, write $\omega = \sum_{i=1}^{n-1} a_i\theta^i$ in some local O.N.B. of 1-forms $\{\theta^i\}$

Then, $Ric^\Sigma(\omega^\#,\omega^\#) = \sum_{i,k} R^\Sigma_{ik}a^ia^k = -\sum_{i,j,k} h_i^j h_{jk}a^ia^k = -|A(\omega^\#)|^2$

Now, we compute

(*) Note: $\frac{1}{2}\Delta(|\omega|^2) = |\omega|\Delta|\omega| + |\nabla|\omega||^2$

$$\begin{aligned}
|\omega|\,L(|\omega|) &= |\omega|\left(\Delta|\omega| + |A|^2|\omega|\right) \\
&= |\omega|\Delta|\omega| + |A|^2|\omega|^2 \\
&\overset{(*)}{=} \underbrace{\frac{1}{2}\Delta|\omega|^2}_{\text{Bôchner}} - |\nabla|\omega||^2 + |A|^2|\omega|^2 \\
&= |\nabla\omega|^2 + \underbrace{Ric^\Sigma(\omega^\#,\omega^\#)}_{-|A(\omega^\#)|^2} - |\nabla|\omega||^2 + |A|^2|\omega|^2 \\
&= \underbrace{|\nabla\omega|^2 - |\nabla|\omega||^2}_{\geq 0 \text{ by Kato's ineq.}} + \underbrace{|A|^2|\omega|^2 - |A(\omega^\#)|^2}_{\geq 0 \text{ by Cauchy-Schwarz}}
\end{aligned}$$

We want to squeeze out a bit more from the first term.

Enchanced Kato's ineq: $|\nabla \omega|^2 - |\nabla|\omega||^2 \geq \frac{1}{n-2}|\nabla \omega|^2$

(for harmonic ω) $n \geq 3$

Reason: (This is just an algebraic lemma)

locally, $\omega = \sum_i a_i \theta^i$.

i.e. $(a_{i;j})$ is symm. trace-free $(n-1)\times(n-1)$ matrix

ω harmonic $\iff \begin{cases} d\omega = 0 \\ \delta\omega = 0 \end{cases} \iff \begin{cases} a_{i;j} = a_{j;i} \\ \sum_i a_{i;i} = 0 \end{cases}$

WLOG, at $p \in \Sigma$, assume $a_1(p) = |\omega|$, $a_i(p) = 0$ for $i \geq 2$

i.e. $\theta^1 = \frac{\omega}{|\omega|}$ at p.

Claim: $|\nabla|\omega||^2 = \sum_k a_{1;k}^2$ at p

Pf: $\nabla|\omega|^2 = 2|\omega|\nabla(|\omega|)$

$$\Rightarrow \underbrace{4|\omega|^2}_{4a_1^2}|\nabla|\omega||^2 = |\nabla|\omega|^2|^2 = \sum_k (\overbrace{|\omega|^2}^{\sum_i a_i^2})_{;k}^2$$

$$= \sum_k \Big(2\sum_i a_i a_{i;k}\Big)^2 = 4a_1^2 \sum_k a_{1;k}^2 \qquad \blacksquare$$

So, $|\nabla|\omega||^2 = \sum_{k=1}^{n-1} a_{1;k}^2 = a_{1;1}^2 + \sum_{k\neq 1} a_{1;k}^2$

$$a_{1;1}^2 = \Big(-\sum_{k\neq 1} a_{k;k}\Big)^2 \leq (n-2)\sum_{k\neq 1} a_{k;k}^2$$

Altogether,

$$\Big(1+\frac{1}{n-2}\Big)|\nabla|\omega||^2 \leq \sum_k a_{1;k}^2 + \sum_{k\neq 1} a_{1;k}^2 + \sum_{k\neq 1} a_{k;k}^2 \overset{(\#)}{\leq} |\nabla\omega|^2$$

(#) $\nabla\omega = (a_{i;j})$

$|\nabla\omega|^2 = \sum_{i,j} a_{i;j}^2$

$$(a_{i;j}) = \begin{pmatrix} a_{1;1} & \cdots & a_{1;k} & \cdots \\ \vdots & & & \\ a_{k;1} & & a_{i;j>1} & \\ \vdots & & & \end{pmatrix}$$

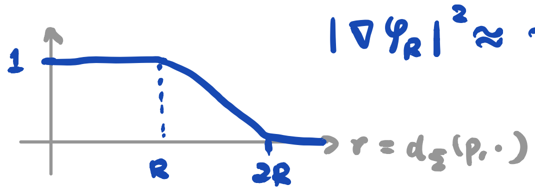
This finishes step 2.

Step 3: Weighted stability & a cutoff argument.

Step 1 & 2 imply $\forall \varphi \in C_c^\infty(\Sigma)$,

$$\frac{1}{n-2}\int_\Sigma |\nabla\omega|^2\varphi^2 \underset{\text{Step 2}}{\leq} \int_\Sigma (|\omega| L|\omega|)\varphi^2 \underset{\text{Step 1}}{\leq} \int_\Sigma |\omega|^2|\nabla\varphi|^2$$

Take $\varphi = \varphi_R$ cutoff fcn



$|\nabla\varphi_R|^2 \approx \frac{1}{R^2}$ in $B^\Sigma_{2R}\setminus B^\Sigma_R$

(Note: NO NEED for logarithmic cutoff trick)

We have

$$\frac{1}{n-2}\int_\Sigma |\nabla\omega|^2\varphi_R^2 \leq \int_\Sigma |\omega|^2|\nabla\varphi_R|^2 \leq \int_{B^\Sigma_{2R}\setminus B^\Sigma_R} |\omega|^2\cdot\frac{C}{R^2} \leq \frac{C}{R^2}\overbrace{\int_\Sigma |\omega|^2}^{<+\infty}$$

As $R\to\infty$, this implies $\nabla\omega \equiv 0$, ie. ω is a parallel 1-form

$\Rightarrow$ $|\omega| \equiv$ const. $= 0$ ($\because \int_\Sigma |\omega|^2 < +\infty$ & Σ has infinite area.)

$\blacksquare$

Now, we proceed to prove:

1-sided result by Ros

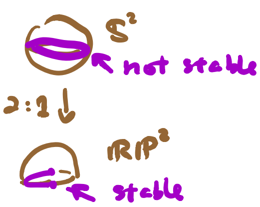
Thm (FCS '80) Any complete, 2-sided, stable min. surface in $\mathbb{R}^3$ is a plane.

Proof: Let $\Sigma^2 \subseteq \mathbb{R}^3$ be the stable min. surface.

Claim 1: (Covering stability)

The universal cover $\tilde{\Sigma} \to \Sigma \hookrightarrow \mathbb{R}^3$ is still a stable min. immersion. [Nontrivial, eg $\mathbb{RP}^2 \subseteq \mathbb{RP}^3$]

or



Claim 2: $\tilde{\Sigma} \approx \mathbb{C}$ conformally.

We prove these two claims first.

<u>Proof of Claim 1</u>:

1st Dirichlet eigenvalue on Ω

- Σ stable $\Leftrightarrow$ $\lambda_1(-L,\Omega) \geq 0 \quad \forall\, \Omega \subset\subset \Sigma$

"domain monotonicity"

- $\lambda_1(-L,\Omega_1) > \lambda_1(-L,\Omega_2)$ $\Rightarrow$ $\lambda_1(-L,\Omega) > 0 \quad \forall\, \Omega \subset\subset \Sigma$

  where $\Omega_1 \subsetneq \Omega_2 \subset\subset \Sigma$

- "Fredholm alternative" $\Rightarrow$ $\exists!$ solution $u_R > 0$ st.

$$\begin{cases} L u_R = 0 & \text{in } \Omega = B_R^{\Sigma} \subset\subset \Sigma. \\ u_R|_{\partial\Omega} = 1 \end{cases}$$

based at $p = 0$.

- Set $v_R := \dfrac{u_R}{u_R(0)}$ where $0 = p \in \Sigma$ is some fixed pt.

THEN, Harnack ineq. & elliptic theory $\Rightarrow$ $\|v_R\|_{C^3(K)} \leq C(K)$

on any $K \subset\subset \Sigma$.

- By Arzela-Ascoli $\Rightarrow$ $\exists$ subseq $R_i \to +\infty$ s.t.

$$v_{R_i} \xrightarrow{C^2 \text{ on cpt subsets}} v > 0$$

s.t. $\begin{cases} Lv = 0 & \text{in } \Sigma \\ v(0) = 1 & (\text{ie } v \text{ is non-trivial}) \end{cases}$

i.e. $\exists$ positive Jacobi field on the entire Σ.

- Lift v from Σ to $\tilde{\Sigma}$, ie. $w := v \circ \pi$ where $\pi : \tilde{\Sigma} \to \Sigma$

$\Rightarrow$ $w \in C^\infty(\tilde{\Sigma})$ and $L^{\tilde{\Sigma}} w = 0$ (ie. $\tilde{\Delta} w + |\tilde{A}|^2 w = 0$)

$w > 0$

where $L^{\tilde{\Sigma}} := \Delta_{\tilde{\Sigma}} + |\tilde{A}|^2$

• Claim: $\tilde\Sigma$ is stable, ie $\displaystyle\int_{\tilde\Sigma} |\tilde A|^2\varphi^2 \le \int_{\tilde\Sigma} |\tilde\nabla\varphi|^2 \quad \forall \varphi\in C_c^\infty(\tilde\Sigma)$

Reason: $w>0$ on $\tilde\Sigma$ $\Rightarrow$ $\log w$ is well-defined on $\tilde\Sigma$.

$$\tilde\Delta(\log w) = \widetilde{\mathrm{div}}\left(\frac{\tilde\nabla w}{w}\right) = \frac{\tilde\Delta w}{w} - \frac{|\tilde\nabla w|^2}{w^2} = -|\tilde A|^2 - |\tilde\nabla(\log w)|^2$$

Multiply by $\varphi^2\in C_c^\infty(\tilde\Sigma)$, integrate.

$$\int_{\tilde\Sigma}\Big(|\tilde A|^2 + \underbrace{|\tilde\nabla \log w|^2}_{\text{cancels}}\Big)\varphi^2 = -\int_{\tilde\Sigma} (\tilde\Delta\log w)\varphi^2$$

$$\overset{\text{I.B.P.}}{=} 2\int_{\tilde\Sigma} \varphi\,\tilde\nabla\varphi\cdot\tilde\nabla\log w$$

$$\le \int_{\tilde\Sigma} |\tilde\nabla\varphi|^2 + \underbrace{|\tilde\nabla\log w|^2\varphi^2}_{\text{cancels}}$$

Proof of Claim 2: $\tilde\Sigma$ simply connected $\overset{\text{uniformization}}{\Longrightarrow}$ $\tilde\Sigma \approx \mathbb{C}, \underbrace{\mathbb{D}}_{\text{rule this out}}, \cancel{S^2}$ $\because$ $\tilde\Sigma$ is non-cpt

$\because$ $\mathbb{D}$ has a non-zero harmonic 1-form, say $\omega = dx$, L^2 w.r.t flat metric

BUT: harmonicity of ω & $\int|\omega|^2<+\infty$ are conformally invariant.

By L^2-vanishing theorem, $\tilde\Sigma \not\approx \mathbb{D}$. So, $\tilde\Sigma\approx\mathbb{C}$. ∎

Now, $w>0$ on $\tilde\Sigma$, and $\tilde\Delta w = -|\tilde A|^2 w \le 0$

ie. w is a positive superharmonic fcn on $\mathbb{C}$

Since $\mathbb{C}$ is parabolic, $w \equiv$ const. >0 $\Rightarrow$ $|\tilde A|^2 \equiv 0$, ie flat.

(set $u=\log w$, then $\int \eta^2|\nabla u|^2 \le \int -\Delta u\cdot\eta^2$) ∎

Q: What about higher dim (but co-dim 1)?

We have at least the following result:

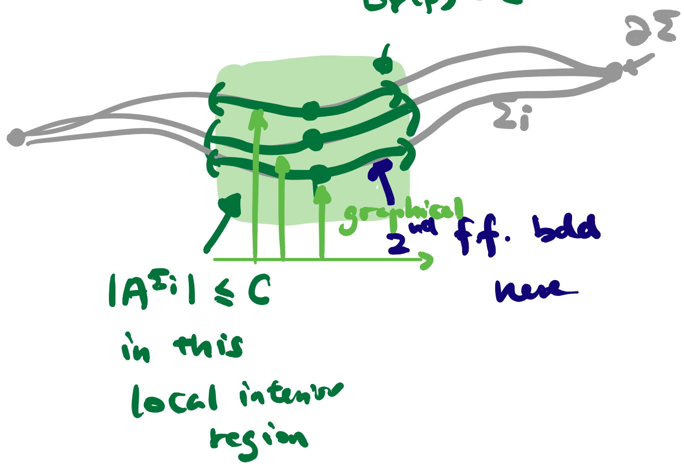
Thm: (Schoen-Simon-Yau '75)

Let $3 \leq n \leq 6$. Suppose $\Sigma^{n-1} \subseteq \mathbb{R}^n$ is a complete, stable, 2-sided min. immersed hypersurface s.t. it has Euclidean volume growth

$$\exists C > 0 \text{ s.t. } |\Sigma \cap B_R(0)| \leq C R^{n-1} \quad \forall R > 0.$$

Then, Σ is a flat hyperplane.

Key Idea: "Curvature Estimates"



Σ stable

$$\overset{?}{\Rightarrow} \quad |A^\Sigma|^2(p) \leq \frac{C}{d_\Sigma^2(p, \partial\Sigma)} \quad \forall p \in \Sigma$$

$\Downarrow$ Bernstein Thm. (think of $\partial\Sigma \to \infty$)

$\Downarrow$ "compactness theorems"

Recall: $A \approx dN$

$$|A| \leq C \Rightarrow \|N\|_{C^1} \leq C$$